Exact volume of hyperbolic 2–bridge links

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Abstract. W. Thurston suggested a method for computing hyperbolic volume of hyperbolic 3–manifolds, based on a triangulation of the manifold. The method was implemented by J. Weeks in the program SnapPea, which produces a decimal approximation as a result. For hyperbolic 2-bridge links, we give formulae that allow one to find the exact volume, i.e. to construct a polynomial and to find volume as an analytic function of one of its roots. The computation is performed directly from a reduced, alternating link diagram.

Key words and phrases: Links, Knots, Hyperbolic, Complement, Volume, Geometric Structure

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1. Introduction

The purpose of this note is to calculate the volume of hyperbolic 2-bridged links exactly, *i.e.* to construct such a polynomial, that the volume can be expressed as an analytic function of one of its roots. It turns out that one can do it directly from a reduced, alternating diagram of a 2-bridged link. The suggested constructive process amounts to assigning labels to the link diagram, counting twists and bigons, and substituting the corresponding labels and numbers in the given formulas. This idea emerged from an example, considered by M. Thistlethwaite.

This work was motivated by two questions. The first one concerns computing the volume of a hyperbolic 3-manifold exactly. W. Thurston suggested a method for computing the volume, based on a triangulation of a manifold ([9]). If done by hand, the process becomes tedious even for manifolds with just several tetrahedra. The method was implemented by J. Weeks in the program SnapPea ([10]), which produces a decimal approximation as a result. The program Snap ([2]), intended for exact calculations, followed. It approximates the hyperbolic structure to a high precision, and then makes an intelligent guess of the corresponding algebraic numbers (from which the volume can be computed). For hyperbolic 2-bridged links, we suggest a simple alternative.

The second question concerns relating diagrammatic properties of a link to the geometry of its complement, and, in particular, to its hyperbolic volume. In this spirit, estimates for a volume of various families of links were previously obtained. For example, M. Lackenby showed in [5] that the hyperbolic volume of an alternating link has upper and lower bounds as functions of the number of twist regions of a reduced, alternating diagram. D. Futer, E. Kalfagianni and J. Purcell extended these results to highly twisted knots ([3]) and to sums of alternating tangles ([4]). We do not provide any bounds, but the suggested calculation of the exact volume is based solely on the layout of a reduced link diagram.

2. Obtaining exact edge and crossing labels

In [8], an alternative method for computing hyperbolic structure of a link complement is described. The key idea of the method is to use isometries of ideal polygons arising from

the regions of a link diagram. The method parameterizes the horoball pattern obtained by lifting cusp neighborhoods to the universal cover $\mathbb{H}^3$ using complex numbers, called edge and crossing labels. According to their names, crossing labels are assigned to crossings of a link projection, and edge labels are assigned to edges of the regions.

Consider an alternating, reduced diagram D of a hyperbolic 2–bridge link L with k twists, where the leftmost twist has n_1 crossings, the twist next to it has n_2 crossings, an so on up to n_k . We will call the leftmost twist - the first, the next twist to the right - the second, etc. Note that there always exists a reduced, alternating diagram of a 2-bridge link such that the first and the last twists have at least two crossing each, so we may assume that $n_1 > 1$ and $n_k > 1$.

We assume that horospherical cross-sections of the cusps of $S^3 \backslash L$ have been chosen so that a (geodesic) meridian curve on the cross-sectional torus has length 1. The preimage of each cross-sectional torus in the universal cover $\mathbb{H}^3$ is a union of horospheres, and we specify a complex affine structure on each horosphere by declaring that meridional translation corresponds to the real number 1. Finally, we assume that coordinates in $\mathbb{H}^3$ are chosen so that one of the horospheres is the Euclidean plane of (Euclidean) height 1 above the xy–plane. With these conventions, a crossing label contains the geometric information about a preimage of an arc from an overpass to an underpass of the crossing. An edge label contains information about a preimage of the corresponding arc on the boundary torus. Further details and rigorous definitions of the labels can be found in [8].

Endow D with a labeling scheme $\mathcal{L}$ as follows. Denote w_1 the leftmost crossing label of the first horizontal twist (and hence all crossing labels of that twist), w_2 the leftmost crossing label of the second twist (and hence all crossing labels of that twist), and so on up to w_k. Let us label the edges of the regions adjacent to twists as shown on the Fig. 1. In addition, let us choose an orientation of the link, and let $\epsilon_j = 1$, if two strands in the j^{th} twist are oriented coherently, and $\epsilon_j = -1$ otherwise.

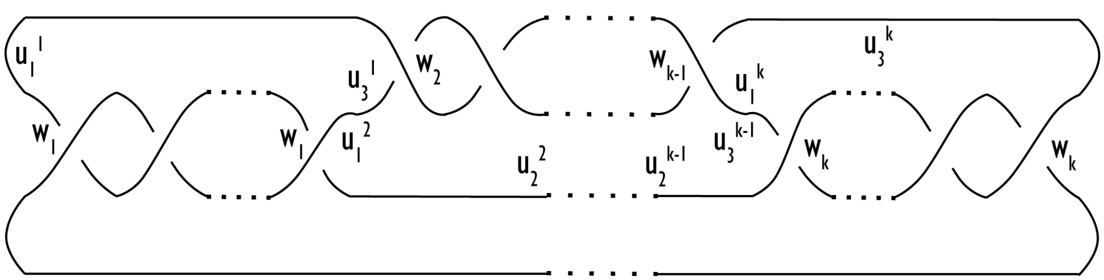


Fig. 1

The layout of a reduced, alternating diagram of a 2-bridge link allows us to find all the labels solely from the label w_1 . This can be done recursively, using regions of a diagram and proceeding from left to right. The rightmost region yields an extra relation, which gives a polynomial in w_1 . Therefore, w_1 can be found exactly, and all other labels can be expressed in the terms of w_1 .

The recursive formulae for the labels are a bit unwieldy, but are obtained in a straightforward manner from the region equations given in [8], as we demonstrate in the next proposition. They involve Fibonacci type polynomials defined for the j^{th} twist as follows: $f_0 = 0$, $f_1 = 1$, and $f_{m+1} = f_m - \epsilon_j w_j f_{m-1}$ for all natural m . The closed formulae for the labels can be obtained as well, though they are not of interest to us in this paper.

Proposition 2.1. Let D be an alternating, reduced diagram of a hyperbolic 2-bridge

link, endowed with the labeling scheme $\mathcal{L}$ as above. Suppose the j^{th} twist has n crossings. Then the labels u_3^j, u_1^j, and w_{j+1} can be found from w_j as follows:

(i) for $j=1$, $u_1^1 = u_3^1 = \frac{\epsilon_1 w_1 f_{n-1}}{f_n}$ and $w_2 = -\frac{(\epsilon_1 w_1)^n}{\epsilon_2 f_n^2}$;

(ii) for $1<j<k$ and $n>1$, $u_1^j = u_3^{j-1} + (-1)^{j-1}$, $u_3^j = \frac{w_j(\epsilon_j u_1^j f_{n-1}(-1)^{j+1} - w_j f_{n-2})}{u_1^j f_n + (-1)^j \epsilon_j w_j f_{n-1}}$, and $w_{j+1} = \frac{\epsilon_{j-1}(w_j)^n w_{j-1}}{\epsilon_{j+1}(u_1^j f_n + (-1)^j \epsilon_j w_j f_{n-1})^2}$;

(iii) for $1<j<k$ and $n=1$, $u_1^j = u_3^{j-1} + (-1)^{j-1}$, $u_3^j = \frac{\epsilon_j w_j}{u_1^j}$, $w_{j+1} = \frac{\epsilon_{j-1}\epsilon_j\epsilon_{j+1} w_{j-1} w_j}{(u_1^j)^2}$;

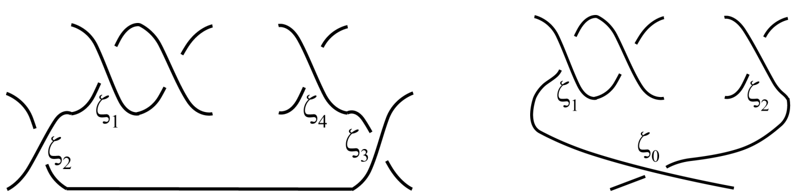
(iv) for $j=k$, $u_3^k = \frac{(-1)^{k+1}\epsilon_k w_k f_{n-1}}{f_n}$ and $u_k^1 = u_3^{k-1} + (-1)^{k-1}$.

Proof. Let R_j be a region of the diagram D, adjacent to the j^{th} twist with n crossings. The region R_j gives a rise to a disk Δ_j, whose boundary consists of sub-arcs on the peripheral torus travelling between adjacent crossings incident to R_j, and arcs travelling between the underpass and the overpass at crossings of R_j. Denote the corresponding ideal polygon in the cover $\mathbb{H}^3$ by $\tilde{\Delta}_j$. Recall from [8] that the shape parameter assigned to the preimage of an arc from an overpass to an underpass is, up to sign, the quotient of the label at that crossing by the product of the two incident edge labels.

Without loss of generality, let us choose such a checkerboard coloring of the regions of the diagram D so that the bigons in the first twist are black. Recall that the edge labels are -1 outside a white bigon, and 1 outside a black bigon.

For every twist, there are two possible situations, namely the number of crossings $n=1$, and $n>1$. Suppose first that $n>1$. Then for $1<j<k$ (Fig. 2(i)), the shape parameters for R_j are

$$\zeta_1 = (-1)^{j+1}\frac{\epsilon_j w_j}{u_1^j}, \quad \zeta_2 = \frac{\epsilon_{j-1} w_{j-1}}{u_1^j u_2^j}, \quad \zeta_3 = \frac{\epsilon_{j+1} w_{j+1}}{u_3^j u_2^j}, \quad \text{and} \quad \zeta_4 = (-1)^{j+1}\frac{\epsilon_j w_j}{u_3^j}.$$



(i) $1<j<k$ *(ii)* $j=0$ *or* $j=k$

Fig. 2

For the first and last regions the situation is slightly different (Fig. 2(ii)):

$$\zeta_1 = \frac{\epsilon_1 w_1}{u_1^1}, \quad \zeta_0 = \frac{\epsilon_2 w_2}{u_3^1 u_1^1}, \text{ and } \quad \zeta_2 = \frac{\epsilon_1 w_1}{u_3^1} \quad \text{for } R_1;$$

$$\zeta_1 = (-1)^{k+1}\frac{\epsilon_k w_k}{u_3^k}, \quad \zeta_0 = \frac{\epsilon_{k-1} w_{k-1}}{u_1^k u_3^k}, \quad \text{and} \quad \zeta_2 = (-1)^{k+1}\frac{\epsilon_k w_k}{u_1^k} \quad \text{for } R_k.$$

All the other shape parameters of R_j are $\epsilon_j w_j$ for all j.

Let $Z_i = \begin{bmatrix} 0 & -\zeta_i \\ 1 & -1 \end{bmatrix}$, $0 \le i \le 4$, and $\mathrm{W} = \begin{bmatrix} 0 & -\epsilon_j w_j \\ 1 & -1 \end{bmatrix}$. Denote c_m the element in row 2, column 1 of the product matrix $Z_3 Z_2 Z_1 W^{m-2} Z_4$.

First suppose $1<j<k$, *i.e.* the j^{th} twist is not the first or the last one. Using mathematical induction, one can prove that

$$c_m = (-1)^{m-1} f_m + (-1)^m f_{m-1}\zeta_1 + (-1)^m f_m \zeta_2 \quad \text{for all natural} \quad m > 1 \ .$$

The product $Z_3Z_2Z_1W^{n-2}Z_4$ corresponds to the composition of hyperbolic isometries, rotating the polygon $\tilde{\Delta}_j$. Since the polygon closes up, this composition is 1, whence $Z_3Z_2Z_1W^{n-2}Z_4$ is a scalar multiple of the identity matrix. Therefore, $c_n = 0$. This together with the above equality for c_m implies $\zeta_2 = \frac{f_n - f_{n-1}\zeta_1}{f_n}$. Substituting the shape parameters, we obtain

$$u_2^j = \frac{\epsilon_{j-1} w_{j-1} f_n}{u_1^j f_n + \epsilon_j w_j f_{n-1} (-1)^j} \ .$$

Similarly, the product $Z_2Z_1W^{n-2}Z_4Z_3$ yields

$$(-1)^{n-1} f_n + (-1)^{n-1}\zeta_1\zeta_4 f_{n-2} + (-1)^n(\zeta_1 + \zeta_4) f_{n-1} = 0 \ , \quad \text{whence} \quad \zeta_4 = \frac{f_n - f_{n-1}\zeta_1}{f_{n-1} - f_{n-2}\zeta_1} \ ,$$

and the equality for u_3^j stated in (ii) follows.

Using the symmetry of the region R_j , we can substitute ζ_1 by ζ_4 , and ζ_2 by ζ_3 in the first equality proved by induction. Then

$$\zeta_3 = \frac{f_n - f_{n-1}\zeta_4}{f_n} = 1 - \frac{f_{n-1}}{f_n} \cdot \frac{f_n - f_{n-1}\zeta_1}{f_{n-1} - f_{n-2}\zeta_1} \ .$$

Note that
$f_{m+1}^2 - f_{m+2} f_m = (f_m - \epsilon_j w_j f_{m-1}) f_{m+1} - (f_{m+1} - \epsilon_j w_j f_m) f_m = \epsilon_j w_j (f_m^2 - f_{m+1} f_{m-1})$ for every natural $m > 1$, and therefore, by induction, $(\epsilon_j w_j)^{m-1} = f_m^2 - f_{m+1} f_{m-1}$. So the relation for ζ_3 becomes $\zeta_3 = \frac{w_j^{n-1}\zeta_1}{f_n(f_{n-1} - f_{n-2}\zeta_1)}$. Substituting the shape parameters and the above formulae for u_2^j and u_3^j , we obtain the relation for w_{j+1} stated in (ii).

For the first and last regions ($j = 1$ and $j = k$), the matrix product $Z_0Z_1W^{n-2}Z_2$ and the symmetry of the region imply $\zeta_1 = \zeta_2 = \frac{f_n}{f_{n-1}}$. The relations (i) and (iv) follow.

In a similar fashion, the product $W^{n-2}Z_2Z_0Z_1$ yields

$$\zeta_0 = \frac{-\zeta_2 f_{n-2} + f_{n-1}}{f_{n-1}} = -\frac{f_{n-1}^2 - f_{n-2} f_n}{f_{n-1}^2} = -\frac{(\epsilon_j w_j)^{n-2}}{f_{n-1}^2} \ ,$$

and the folmula for w_2 stated in (i) follows.

In the case of $n = 1$, R_j is 3–sided, and all the shape parameters are equal to 1. This implies (iii), completing the proof. □

Remark 2.2. In the proof of the Proposition 2.1, the relation $\zeta_2 = \frac{f_n}{f_{n-1}}$ for the last region yields an extra equation

$$u_1^k f_n + (-1)^k \epsilon_k w_k f_{n-1} = 0 \ ,$$

where $n > 1$ is the number of crossings in the last twist. This equation together with the recursive formulae from Proposition 2.1 describe a constructive process of obtaining a polynomial from the link diagram, for which w_1 is a root. The polynomial has several complex roots. The root that corresponds to the geometric structure is the one that maximizes hyperbolic volume.

3. Shapes of ideal tetrahedra

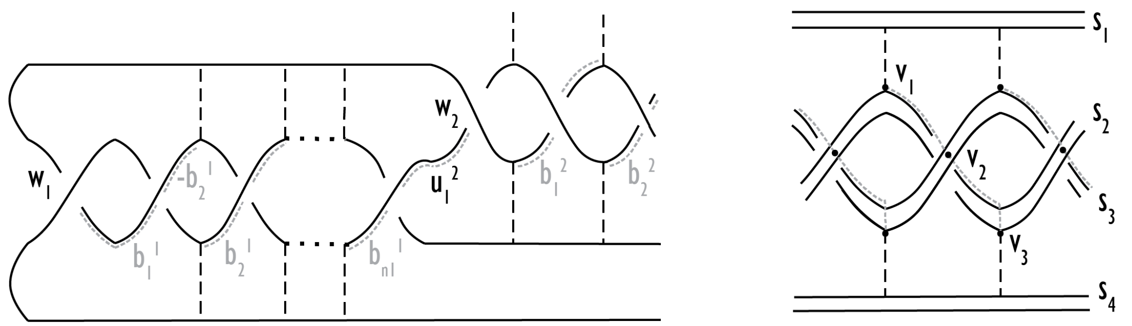
According to the Sakuma-Weeks description ([7]) of the canonical cell decomposition, there are $2(n_1 + n_2 + \ldots + n_k - 3)$ tetrahedra occurring in isometric pairs in the complement

$S^3\backslash L$ of a hyperbolic 2-bridge link L. In this section, we give simple formulas that allow one to find the tetrahedra shapes from the labels of a reduced, alternating diagram D of L.

Suppose D is endowed with a labeling scheme $\mathcal{L}$ as above. We will add more labels, and call the resulting scheme $\mathcal{L}_1$. On Fig. 3(i), the vertical dotted lines represent geodesics in the complement $S^3\backslash L$ that correspond to edges of the canonical cell decomposition. Every b_i^j, except for the first and last ones (b_1^1 and $b_{n_k}^k$), is the edge label corresponding to a (directed) Euclidean line segment on the boundary torus between such a geodesic and the geodesic for the next to the right crossing arc in the twist. The first one, b_1^1, corresponds to the segment from the first (leftmost) crossing arc to the second crossing arc. The last one, $b_{n_k}^k$, corresponds to the segment between the last crossing arc and the one preceding it. In our notation the upper index indicates the adjacent twist.

Vertical dotted lines subdivide the diagram into parts. We will refer to these parts as to "levels". Examining the Sakuma and Weeks description, we see that there are two isometric tetrahedra on every level. We will say that an ideal tetrahedron T is adjacent to a crossing, if an arc from the overpass to the underpass of this crossing is (homotopic to) a truncated edge of T. We will also say that T is adjacent to a twist, if T is adjacent to one of the crossings of this twist.

The following theorem gives expressions for the tetrahedra shapes in terms of the labels.



(i) Decomposition of 2–bridge link complement *(ii) Vertices of a cross-section*

Fig. 3

Theorem 3.1. Let D be the reduced, alternating diagram of a hyperbolic 2-bridged link L with a labeling scheme $\mathcal{L}_1$ as above. In the canonical cell decomposition of $S^3\backslash L$, a tetrahedron shape is z if $Arg(z) > 0$, and $1/z$ otherwise, where z is a ratio of the form

(i) $\frac{b_{n_k-1}^k-1}{b_{n_k}^k}$ for a pair of tetrahedra adjacent to the last crossing of the last twist;

(ii) $-\frac{b_{n_j}^j}{u_1^{j+1}}$ for a pair of tetrahedra adjacent to the j^{th} and $(j+1)^{\text{th}}$ twists, $n_j > 1$;

(iii) $-\frac{u_3^j}{b_1^{j+1}}$ for a pair of tetrahedra adjacent to the j^{th} and $(j+1)^{\text{th}}$ twists, $n_j = 1$;

(iv) $\frac{b_i^j}{b_{i+1}^j}$ for all other pairs of tetrahedra.

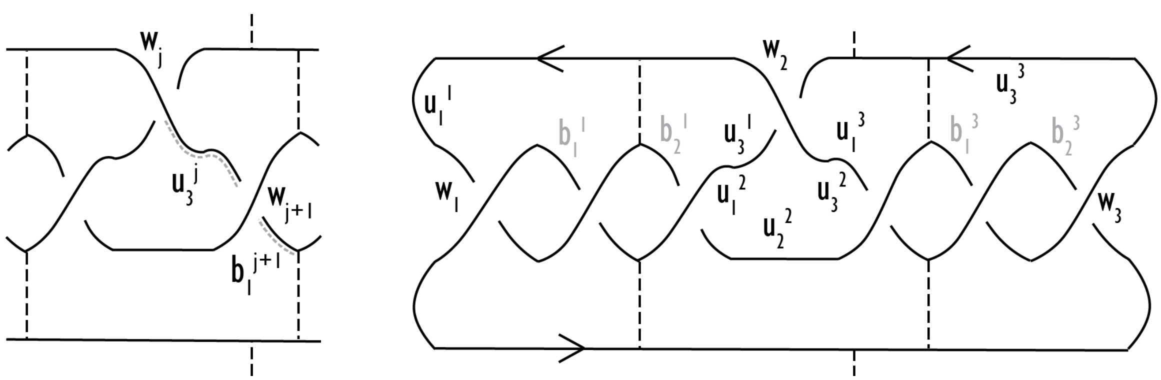
Proof. Denote the strands of the diagram D by s_1, s_2, s_3, s_4. Consider an ideal tetrahedron in the canonical cell decomposition of $S^3\backslash L$. Inspection of the Sakuma-Weeks description ([7]) shows that each Euclidean cusp cross-section on a (thickened) strand s_i has its three vertices v_1, v_2, v_3 on geodesics joining s_i with each of the other three strands. Fig. 3(ii) illustrates this situation for a tetrahedron adjacent to a twist with more

than one crossing. Grey dotted arcs from v_1 to v_2 and from v_2 to v_3 along the boundary torus are edges of the cross-section. Fig. 3(i) extends this picture, showing pairs of edges for cross-sections of multiple tetrahedra.

We have two cases to consider, namely, when a pair of isometric tetrahedra is adjacent just to one twist, and when it is adjacent to two. In the first case, the complex translations corresponding to the edges of a Euclidean cusp cross-section are the edge labels b_{i+1} and $-b_i$ (the latter due to the symmetry of the diagram near a twist), unless tetrahedra are adjacent to the last crossing of the last twist. For that last crossing, the complex translations are $1 - b^k_{n_k - 1}$ and $b^k_{n_k}$. In the second case, i.e. at the levels, where the diagram goes from one twist into another, the situation is slightly different. The complex translations along the edges of a Euclidean cross-section correspond to the complex numbers u^j_3 and b^{j+1}_1 if the j^{th} twist has just one crossing (Fig. 4(i)), and to $b^j_{n_j}$ and u^{j+1}_1 otherwise (as shown on Fig. 3(i)).

From the layout of the diagram, we see that some of crossings are between the strands s_1 and s_2, while all others are between s_2 and s_3. For the first type of crossings, we can write z as suggested in (i)-(iv), and then shapes for the corresponding pair of isometric tetrahedra become the usual z, $1-\frac{1}{z}$, $\frac{1}{1-z}$. The extra minus before the ratios in (i)-(iv) guarantees that the argument of a shape corresponds to the interior dihedral angle of the tetrahedron (not an exterior one). For the second type of crossing, $\frac{1}{z}$ gives a corresponding shape. Note that every shape has a positive imaginary part, while its reciprocal has a negative one. □

It is intended to express shapes of all tetrahedra in terms of just one label, w_1, which is a root of the polynomial (Remark 2.2). With this purpose in mind, we give the following Lemma.



(i) Twist with one crossing *(ii) Knot (3, 1, 3)*

Fig. 4

Lemma 3.2. Let D be a reduced, alternating diagram of a hyperbolic 2-bridged link, endowed with a labeling scheme $\mathcal{L}_1$ as above. Then every b^j_i can be found as follows:

(i) $b^1_1 = 1$;

(ii) $b^k_{n_k} = (-1)^{k+1}$;

(iii) for all other b^i_j, $b^j_{i+1} = (-1)^{j+1} - \frac{\epsilon_j w_j}{b^j_i}$ and $b^j_1 = (-1)^{j+1} - \frac{\epsilon_j w_j}{u^j_1}$.

Proof. In a region adjacent to the j^{th} twist, the edge label along each bigon is $(-1)^{j+1}$. Therefore, $b^1_1 = 1$ and $b^k_{n_k} = (-1)^{k+1}$. To find all other b^j_i, note that the vertical dotted geodesics split the j^{th} region into triangles. All the shape parameters in triangular regions

are 1, and equal the quotient of $\epsilon_j w_j$ by the product of two incident edge labels. From this we obtain (iii). $\square$

Corollary 3.1.1. A shape of every tetrahedron in the canonical cell decomposition of $S^3 \backslash L$ can be written as a rational function of w_1 .

Proof. Theorem 3.1 demonstrates that any tetrahedron shape is a rational function of the labels w_j , u_i^j , and b_i^j . Proposition 2.1 and Lemma 3.2 show that any of this labels, in its turn, can be written as a rational function of w_1 . $\square$

Remark 3.3. The dihedral angles of a tetrahedron with the shape z_p , p from 1 to $2(n_1+n_2+...+n_k-3)$, are $\alpha_p = \mathrm{Arg}\,(z_p)$, $\beta_p = \mathrm{Arg}\left(1-\frac{1}{z_p}\right)$, $\gamma_p = \mathrm{Arg}\left(\frac{1}{1-z_p}\right)$, and the volume is thus $2\sum_{p=1}^{2(n_1+n_2+...+n_k-3)}(\Lambda(\alpha_p)+\Lambda(\beta_p)+\Lambda(\gamma_p))$, where Λ is the Lobachevsky function. Therefore, the volume of $S^3 \backslash L$ can be calculated solely in terms of w_1 .

We proceed with two examples: one of a knot, and one of a link.

Example 3.4. Consider a 2–bridge knot with a Conway code 3 1 3 . There are 4 pairs of isometric tetrahedra in its complement. Fix orientation as on Fig. 4(ii). Note that $\epsilon_1 = \epsilon_3 = -1$, $\epsilon_2 = 1$.

For the region adjacent to the first (leftmost) twist, $f_0 = 0$, $f_1 = f_2 = 1$, $f_3 = f_2 + w_1 f_1$. Therefore, from the Proposition 2.1, $u_1^1 = u_3^1 = -\frac{w_1 f_2}{f_3}$, $w_2 = \frac{(w_1)^3}{(f_3)^2}$. The region adjacent to the second twist has just one crossing, hence we obtain $u_1^2 = u_3^1 - 1$, $u_3^2 = \frac{w_2}{u_1^2}$, $w_3 = \frac{w_1 w_2}{(u_1^2)^2}$. For the region adjacent to the third twist, $f_0 = 0$, $f_1 = f_2 = 1$, $f_3 = f_2 + w_3 f_1$, and therefore, $u_1^3 = u_3^2 + 1$.

Now we can construct the polynomial in w_1 using Remark 2.2 : $u_1^3 f_3 + w_3 f_2 = 0$, which becomes

$$1 + 7w_1 + 18w_1^2 + 19w_1^3 + 6w_1^4 + 2w_1^5 + 4w_1^6 - w_1^7 = 0 \ .$$

The root that gives the geometric structure is

$$w_1 = \frac{-(1196+12\sqrt{177})^{\frac{2}{3}} - 112 + 16(1196+12\sqrt{117})^{\frac{1}{3}} + i\sqrt{3}(1196+12\sqrt{117})^{\frac{2}{3}} - 112i\sqrt{3}}{12(1196+12\sqrt{117})^{\frac{1}{3}}}.$$

This allows to compute all other labels exactly as well.

Let us turn to the edges of the canonical cell decomposition (Fig. 4): $b_1^1 = b_3^2 = 1$, $b_2^1 = 1 + \frac{w_1}{b_1^1}$, and $b_1^3 = 1 + \frac{w_2}{u_1^3}$ by the Lemma 3.2. The shape parameters are $z_1 = -\frac{-b_2^1}{b_1^1}$, $z_2 = -\frac{u_2^1}{b_2^1}$, $z_3 = -\frac{u_3^2}{1-b_1^3}$, $z_4 = -\frac{b_3^1 - 1}{b_2^3}$. If we substitute the decimal approximation of w_1 , we obtain the volume of 5.1379412018734177698 .

Example 3.5. Consider a 2–bridge link with a Conway code 3 2 3 . Fix orientation as on Fig. 5. For this link, there are 5 pairs of isometric tetrahedra in the canonical cell decomposition. Note that $\epsilon_1 = \epsilon_3 = 1$, $\epsilon_2 = -1$.

For the first region, $f_0 = 0$, $f_1 = f_2 = 1$, $f_3 = f_2 - w_1 f_1$. Therefore, from the Proposition 2.1 we obtain $u_1^1 = u_1^3 = \frac{w_1 f_2}{f_3}$, $u_1^2 = u_1^1 - 1$, $w_2 = -\frac{-(w_1)^3}{(f_2)^2}$. For the second region, the Fibonacci-type polynomials f_0 , f_1 , and f_2 are the same, and $u_1^2 = u_3^1 - 1$, $u_3^2 = \frac{w_2 u_1^2 f_1 - f_0 (w_2)^2}{u_1^2 f_2 - w_2 f_1}$, $w_3 = \frac{(w_2)^2 w_1}{(u_1^2 f_2 - w_2 f_1)^2}$, $u_1^3 = u_3^2 + 1$. Using Remark 2.2, we obtain a polynomial:

$$-1 + 7w_1 - 18w_1^2 + 16w_1^3 + 9w_1^9 - 19w_1^5 - 4w_1^6 + 10w_1^7 + 4w_1^8 = 0 \ .$$

A decimal approximation of the root that gives the hyperbolic structure is $w_1 = 0.45899397977032988781 + 0.22363894995478265862511 80 * i$. Now we can go back and compute the other labels.

Turn to the edges of the canonical cell decomposition: $b_1^1 = b_2^3 = 1$, $b_2^1 = 1 - \frac{w_1}{b_1^1}$, $b_1^2 = -1 + \frac{w_2}{u_1^2}$, $b_1^3 = 1 - \frac{w_3}{u_1^3}$ by Remark 3.2. The shapes for five pairs of isometric tetrahedra are $z_1 = -(\frac{b_1^1}{-b_2^1})$, $z_2 = -\frac{b_2^1}{u_1^2}$, $z_3 = -\frac{b_1^2}{u_1^3}$, $z_4 = -\frac{1-b_1^3}{u_3^2}$, $z_5 = -\frac{b_2^3}{b_1^3-1}$. Now we can compute the volume; its decimal approximation is 7.5176898964745685429.

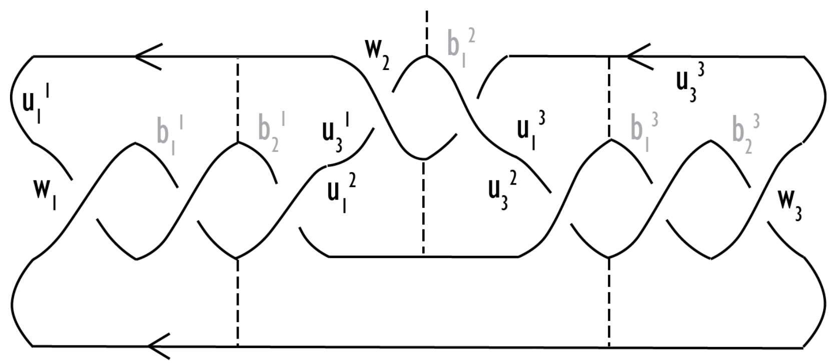


Fig. 5

It would be very interesting if one could generalize the process of obtaining the exact tetrahedra parameters from the diagram beyond the family of two-bridged links.

5. Complex volume

The complex volume is an invariant of hyperbolic manifolds of mixed geometric and algebraic nature. It is a complex number: the real part is a hyperbolic volume, and an imaginary part is Chern-Simons invariant. ([6]).

Similar machinery and formulas from [11] can be used for a computation of the exact complex volume of a hyperbolic 2-bridged link L. In particular, some of our edge labels (e.g., b_i^j) correspond to labels of short edges (in the terminology of [11]) of tetrahedra in the canonical cell decomposition of the complement of L. To see the exact correspondence, faces and edges of every tetrahedron should be located on a reduced, alternating diagram of L, and the gluing pattern should be traced. From these labels, one can compute a flattening of every tetrahedron, and then the complex volume of a complement of L. Such computation would save the step of developing an image of every cusp, allowing one to find the labels from a link diagram instead.

5. Acknowledgments

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